

Branching processes and Koenigs function

O.G. Tchikilev ^a

^a*Institute for High Energy Physics, Protvino, Russia*

Abstract

An explicit solution of non-critical time-homogeneous branching processes is described.

1 Introduction

Branching processes are widely used in high energy physics [1]. For example, the well known Furry-Yule and negative binomial distributions occur in simple branching processes with allowed transition $1 \rightarrow 2$. The use of processes with higher order transitions $1 \rightarrow n$ with $n > 2$ is rare due to the absence of explicit solutions in terms of elementary functions. In this paper we describe the solution for processes with higher order transitions using a recently found recursive procedure for the pure birth branching process[2]. The solution is based on the use of the Koenigs function[3] and the functional Schröder equation[4], sometimes called the Schröder-Koenigs equation (for a detailed description and extensive bibliography see [5,6]). In section 2 we describe the solution[2] for the pure birth branching process. In sections 3 and 4, respectively, the procedures for non-critical branching processes and branching processes with immigration are outlined. The results are summarized in the last section.

2 Solution for the general pure birth branching process

A branching process with continuous evolution parameter t is determined by the rates α_n for the transition (“splitting”) of one particle into n particles with all particles subsequently evolving independently. For a pure birth branching process $\alpha_0 = 0$. The probability distribution $p_n(t)$ for the process having one particle at $t = 0$ with $p_n(0) = \delta_{1n}$ is a solution of the forward Kolmogorov

equation [7,8]

$$\frac{\partial m}{\partial t} = f(x) \frac{\partial m}{\partial x} \quad (1)$$

for the probability generating function

$$m(x, t) = \sum_{n=0}^{\infty} p_n(t) x^n \quad (2)$$

with

$$f(x) = \sum_{n=2}^{\infty} \alpha_n x^n - \alpha x \quad (3)$$

and with $\alpha = \sum \alpha_n$. The Taylor expansion of equation (1) leads to the following system of equations for the probabilities p_n

$$\frac{dp_1}{dt} = -\alpha p_1 \quad , \quad (4)$$

$$\frac{dp_2}{dt} = \alpha_2 p_1 - 2\alpha p_2 \quad , \quad (5)$$

$$\frac{dp_3}{dt} = \alpha_3 p_1 + 2\alpha_2 p_2 - 3\alpha p_3 \quad (6)$$

and for arbitrary n

$$\frac{dp_n}{dt} = \sum_{j=1}^{n-1} j \alpha_{n-j+1} p_j - n \alpha p_n \quad . \quad (7)$$

A simple interpretation of the equation (7) is as follows: let us consider the state with n particles at the moment t . The change in this state is due to the arrival from states with multiplicity lower than n and to the departure to states with higher multiplicity. The arrival rate from the state with j particles is proportional to the number of particles j , to the transition rate (for one particle) to produce $n - j$ new particles, i.e. α_{n-j+1} , and to the population density in the state j , the sum in the equation (7) goes over all states below n . The departure rate is proportional to the total transition rate (for one particle) α , to the number of particles n in this state and to the population density, this explains second term in the equation (7). Formally this system of equations is valid for any initial condition.

Let us recall the recursive solution of the equations (4)-(7) given in [2]: the probability $p_1(t)$ is $p_1 = \exp(-\alpha t)$ and

$$p_n = \sum_{j=1}^n \pi_{jn} p_1^j \quad (8)$$

with the following recursion for the coefficients π_{jn}

$$(n-j)\pi_{jn} = \sum_{l=1}^{n-j} (n-l)b_l \pi_{j(n-l)} \quad . \quad (9)$$

Here $b_l = \alpha_{l+1}/\alpha$ is the relative probability to produce l new particles. The recursion starts from $\pi_{11} = 1$ and the coefficient π_{nn} can be found from the initial condition

$$\pi_{nn} = - \sum_{j=1}^{n-1} \pi_{jn} \quad . \quad (10)$$

For the case with N initial particles with $p_n^{(N)}(0) = \delta_{Nn}$, the first $N-1$ equations are automatically valid since

$$p_1^{(N)} = p_2^{(N)} = \dots = p_{N-1}^{(N)} = 0 \quad (11)$$

and the solution in this case has the following form: $p_N^{(N)} = \exp(-N\alpha t) = p_1^N$ and

$$p_n^{(N)} = \sum_{j=N}^n \pi_{jn}^{(N)} p_1^j \quad (12)$$

with the same recursion as (9) for the coefficients $\pi_{jn}^{(N)}$

$$(n-j)\pi_{jn}^{(N)} = \sum_{l=1}^{n-j} (n-l)b_l \pi_{j(n-l)}^{(N)} \quad . \quad (13)$$

This recursion starts from $\pi_{NN}^{(N)} = 1$ and the coefficient $\pi_{nn}^{(N)}$ can be found from the relation

$$\pi_{nn}^{(N)} = - \sum_{j=N}^{n-1} \pi_{jn}^{(N)} \quad . \quad (14)$$

One can calculate the coefficients $\pi_{jn}^{(N)}$ using the concept of the Koenigs function [2,5,6,9]. For the branching process starting from one particle at $t = 0$ this function is defined as the limit

$$K(x) = \lim_{n \rightarrow \infty} \frac{m(x, nt)}{p_1^n} \quad . \quad (15)$$

$K(x)$ has the following Taylor expansion:

$$K(x) = \sum_{j=1}^{\infty} \kappa_j x^j = \sum_{j=1}^{\infty} \pi_{1j} x^j \quad . \quad (16)$$

For the branching process starting from N particles the Koenigs function is defined analogously:

$$K^{(N)}(x) = K^N(x) = \lim_{n \rightarrow \infty} \frac{m^N(x, nt)}{(p_1^N)^n} = \sum_{j=N}^{\infty} \kappa_j^{(N)} x^j = \sum_{j=N}^{\infty} \pi_{Nj}^{(N)} x^j \quad . \quad (17)$$

The recursion (13) leads to the following recurrence for the coefficients $\kappa_j^{(N)}$, $N = 1, 2, \dots$; $j = N + 1, N + 2, \dots$:

$$(j - N) \kappa_j^{(N)} = \sum_{l=1}^{j-N} (j - l) b_l \kappa_{(j-l)}^{(N)} \quad . \quad (18)$$

Let us denote $\kappa_{x+n}^{(x)} = t_n(x)$, then $t_0(x) = 1$ and $t_n(x)$ is given by the following recursion

$$n t_n(x) = \sum_{l=1}^n (x + n - l) b_l t_{n-l}(x) \quad . \quad (19)$$

It is evident from the equation (19) that the $t_n(x)$ is a polynomial of order n in x . The $\kappa_j^{(N)}$ in terms of $t_n(x)$ is equal to $t_{j-N}(N)$.

The remarkable property of the Koenigs function is that it satisfies the functional Schröder equation:

$$K(m) = p_1 K(x) \quad . \quad (20)$$

It is convenient to introduce the function $Q(x)$ the inverse of the Koenigs function. Then the equation (20) gives the functional relations $m(x, t) =$

$Q(p_1 K(x))$ and $m^N(x, t) = Q^N(p_1 K(x))$. The coefficients of the Taylor expansion for $Q^N(x) = \sum Q_j^{(N)} x^j$ can be found using the following relation (see Appendix B in [10] and references therein):

$$Q_j^{(N)} = \frac{N}{j} \kappa_{-N}^{(-j)} \quad . \quad (21)$$

In terms of $t_n(x)$ it gives

$$Q_j^{(N)} = \frac{N}{j} t_{j-N}(-j) \quad . \quad (22)$$

Finally, comparison of equation (12) with the Taylor expansion in x of the $Q^N(p_1(K(x)))$ leads to the following expression for the coefficients $\pi_{jn}^{(N)}$:

$$\pi_{jn}^{(N)} = Q_j^{(N)} \kappa_n^{(j)} = \frac{N}{j} t_{j-N}(-j) t_{n-j}(j) \quad . \quad (23)$$

3 Solution for non-critical branching processes

In this case the absorption coefficient α_0 is not equal to zero and the function $f(x)$ (3) has an additional term $\alpha_0(1-x)$. Let us denote by β the smallest positive, different from unity root of the equation $f(x) = 0$. Branching processes with $0 < \beta < 1$ are called supercritical branching processes and processes with $1 < \beta$ are called subcritical ones.

The solution for the supercritical branching processes is obtained in the following way. Let us transform m and x to m' and x' by the linear transform

$$x' = \frac{x - \beta}{1 - \beta} \quad , \quad m' = \frac{m - \beta}{1 - \beta} \quad , \quad (24)$$

this transform moves the root β to zero. Then the forward Kolmogorov equation (1) for the process with absorption transforms to the form for the pure birth branching process. Therefore one can introduce the Koenigs function with

$$K\left(\frac{m - \beta}{1 - \beta}\right) = q_1 K\left(\frac{x - \beta}{1 - \beta}\right) \quad , \quad (25)$$

where $q_1 = \exp(-\alpha't)$. This leads to the expression for the $m(x, t)$

$$m(x, t) = \beta + (1 - \beta)Q\left(q_1 K\left(\frac{x - \beta}{1 - \beta}\right)\right) \quad (26)$$

and to the infinite series in q_1 for the probabilities p_n

$$p_n = \beta\delta_{0n} + (1 - \beta) \sum_{l=n}^{\infty} \frac{(-\beta)^{l-n}}{(1 - \beta)^l} \frac{l!}{n!(n-l)!} \sum_{j=n}^l Q_j q_1^j \kappa_l^{(j)} \quad . \quad (27)$$

The solution for the subcritical branching processes ($\beta > 1$) can be obtained in a similar way. In this case the linear transform should move 1 to zero and β to 1, i.e. $x' = (x - 1)/(\beta - 1)$. This leads to the similar expressions

$$m(x, t) = 1 + (\beta - 1)Q\left(q_1 K\left(\frac{x - 1}{\beta - 1}\right)\right) \quad (28)$$

and

$$p_n = \delta_{0n} + (\beta - 1) \sum_{l=n}^{\infty} \frac{(-1)^{l-n}}{(\beta - 1)^l} \frac{l!}{n!(n-l)!} \sum_{j=n}^l Q_j q_1^j \kappa_l^{(j)} \quad . \quad (29)$$

The procedures described above are not suitable for critical branching processes with $\beta = 1$. In simplest case without higher order transitions the probability generating function $m(x, t)$ is a linear fraction with coefficients having linear dependence on t (instead of $\exp(-\alpha t)$ dependence for non-critical processes). In general case with higher order transitions the interplay between t and $\exp(-\alpha t)$ dependences leads to more complex equations.

4 Solution for branching processes with immigration

For the branching processes with immigration there is an additional external source of particles appearing in clusters of j particles with the differential rates β_j ($\sum \beta_j = b$). The generating function for the process starting with zero particles at $t = 0$ can be written [11,12] as

$$M(x, t) = \exp\left(\int_0^t g(m(x, \tau)) d\tau\right) \quad (30)$$

with

$$g(x) = \sum_{i=1}^{\infty} \beta_i x^i - b \quad , \quad (31)$$

where $m(x, \tau)$ is the solution for the underlying branching process without immigration. For the underlying pure birth branching process, equation (30) leads to the following expression

$$M(x, t) = \exp(-bt) \exp\left(\sum_{n=1}^{\infty} v_n(t) x^n\right) \quad (32)$$

with

$$v_n(t) = \sum_{i=1}^n \sum_{j=i}^n \pi_{jn}^{(i)} \frac{1 - p_1^j(t)}{j\alpha} \quad . \quad (33)$$

Let us denote

$$\exp\left(\sum_{n=1}^{\infty} v_n x^n\right) = 1 + \sum_{n=1}^{\infty} V_n x^n \quad , \quad (34)$$

then the final probability $P_n(t) = \exp(-bt)V_n(t)$. The coefficients $V_n(t)$ can be calculated using the recursive relation:

$$nV_n = \sum_{j=1}^n jv_j V_{n-j} \quad . \quad (35)$$

This relation is known in combinatorics and is used, for example, in the study of combinants[13–16].

5 Summary

In this paper we have derived explicit expressions for the probability distributions in various branching processes. Although we have not derived closed expressions for the polynomials $t_n(x)$, the given recursions can serve as a calculational tool both in theoretical and experimental studies.

References

- [1] For a recent review see: R.C. Hwa, *Branching processes in multiparticle production*, in Hadronic Multiparticle Production, edited by P. Carruthers, World Scientific, Singapore, 1988.
- [2] O.G. Tchikilev, *Phys.Lett.* **B471** (2000) 400; erratum, *Phys.Lett.* **B478** (2000) 459.
- [3] G. Koenigs, *Recherches sur les intégrales de certaines équations fonctionnelles*, Ann. Sci. Ecole Norm. Sup. (3)1(1884), Supplément pp. 3-41; *Nouvelles recherches sur les équations fonctionnelles*, ibid. (3)2 (1885) pp. 385-404.
- [4] E. Schröder, *Über unendlich viele Algorithmen zur Auflösung der Gleichungen*, Math. Ann. 2(1870) pp. 317-365; *Über iterierte Funktionen*, ibid. 3(1871) pp. 296-322.
- [5] M. Kuczma, *Functional equations in a single variable*, PWN - Polish Scientific Publishers, Warszawa, 1968.
- [6] M. Kuczma, B. Choczewski and R. Ger, *Iterative functional equations*, Cambridge Univ. Press, Cambridge-New York-New Rochelle- Melbourne-Sydney, 1989.
- [7] N.A. Dmitriev and A.N. Kolmogorov, Dokl. Acad. Nauk SSSR, **56** (1947) 7.
- [8] B.A. Sevastyanov, Uspekhi Math. Nauk, **6** (1951) 47.
- [9] G. Valiron, *Fonctions analytiques*, Presse Universitaire De France, Paris, 1954.
- [10] J.F. Traub, *Iterative methods for the solution of equations*, Chelsea Publishing Company, New York, 1982.
- [11] M.S. Bartlett, *An introduction to stochastic processes with special reference to methods and applications*, Cambridge Univ. Press, Cambridge, 1955.
- [12] P.V. Chliapnikov and O.G. Tchikilev, *Phys.Lett.* **B235** (1990) 347.
- [13] M. Gyulassy and S.K. Kauffmann, *Phys.Rev.Lett.* **40** (1978) 298.
- [14] S.K. Kauffmann and M. Gyulassy, *J.Phys.* **A11** (1978) 1715.
- [15] A.B. Balantekin and J.E. Seger *Phys.Lett.* **B266** (1991) 231.
- [16] S. Hegyi, *Phys.Lett.* **B309** (1993) 443; ibid. **B318** (1993) 642.